The Decision-Making Properties of Discrete Multiple Exponential Bidirectional Associative Memories
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Abstract—A method for modeling the learning of belief combination in evidential reasoning using a neural network is presented. A centralized network composed of multiple exponential bidirectional associative memories (eBAM's) sharing a single output array of neurons is proposed to process the uncertainty management of many pieces of evidence simultaneously. The stability of the proposed multiple eBAM network is proved. The sufficient condition to recall a stored pattern pair is discussed. The most important of all, a majority rule of decision making in presentation of multiple evidence is also found by the study of signal-noise-ratio of multiple eBAM network. A guaranteed stable state condition, i.e., the condition for the fastest recall of a pattern pair, is also studied. The result is coherent with the intuition of reasoning.

I. INTRODUCTION

NEURAL networks have been drawing increasing interest as powerful tools to solve different tasks of artificial intelligence [2], [3], [12]. An associative memory is one type of neural network which essentially is a single functional layer or slab that associates one set of vectors with another set of vectors. Kosko [6] proposed a two-level nonlinear network, bidirectional associative memory (BAM), which extends a one-directional process to a two-directional process. Jeng [5] and Wang [13], respectively, then generalized the concept of storing information in the exponential BAM (eBAM).

Among the problems of evidential reasoning, conflicts caused by sequential programming and partial dependency are pretty hard to be fully resolved [8], [10]. The basic reason is all of the traditional methods for evidential reasoning are developed for two pieces of evidence. Thus, when there are more than two pieces of evidence, conflicts will happen if the combination orders are different [11]. Wang et al. pointed out the importance of simultaneously processing many pieces of evidence [10], and he further proposed a method using multiple BAM structure to handle the demand of combining many evidence at the same time [12]. Because the relationship of evidence and the hypothesis is always referred to be an IF-and-THEN relationship. Hence, this IF-and-THEN format can be easily transformed into numbers which can be stored in memories, more specifically, associative memories. If people intend to evaluate the degree of a piece of evidence supporting a hypothesis, they simply present the evidence to their memory to recall stored information. If there are more than a piece of evidence, then present all of the evidence and see the result of their common output. Since the more evidence support one hypothesis, the result should be drawn closer to this hypothesis. Due to the inherently poor capacity of BAM [5], [13], however, obviously the multiple BAM network would be limited to a foreseeable degree of processing capability. We propose a multiple eBAM network to increase the processing capability of reasoning many evidence. We also discuss the majority rule of decision making for handling many evidence at the same time. The majority rule means that if more than half of the presented evidence support one hypothesis, the result of the belief combination of all of the evidence must be inclined toward this hypothesis. This rule is intuitively in accordance with the human reasoning.

In this paper, we adopt the exponential form and combine it with the multiple BAM structure to enhance the signal-noise-ratio (SNR) of the entire network and, consequently, increase the capacity. We also prove the stability of the multiple eBAM network. A majority factor (k) is determined to indicate what the portion of the total amount of eBAM is necessary to reach the dominant hypothesis. The guaranteed stable state condition of pattern pairs, or the fastest recall condition of pattern pairs, is studied. The simulation result is much more appealing than the previous works.

II. FRAMEWORK OF MULTIPLE EXPONENTIAL BAM’S NETWORK

A. Evolution Equations

As shown in Fig. 1, the multi-eBAM network is constructed with L single eBAM’s which share a common output array of neurons. In each clock, the input vectors are presented at the input array of neurons, respectively. Suppose we are given N training sample pairs to the qth eBAM of the network, which are

$$\{(A_{q1}, B_1), (A_{q2}, B_2), \ldots, (A_{qN}, B_N)\} \quad (1)$$

where

$$A_{qi} = (a_{q1i}, a_{q2i}, \ldots, a_{qiN})$$

$$B_i = (b_{1i}, b_{2i}, \ldots, b_{pi}).$$

Let $X_i$ and $Y_i$ be the bipolar mode of the training pattern pairs, $A_{qi}$ and $B_i$, respectively. That is, $X_i \in \{-1, 1\}^n$ and $Y_i \in \{-1, 1\}^p$. Thus, we use the following evolution equations

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in the recall process of the multi-e BAM network

\[ y_k = \begin{cases} 
1 & \text{if } \sum_{i=1}^L \sum_{j=1}^N y_{ik} b_x^i X_i x_j \geq 0 \\
-1 & \text{if } \sum_{i=1}^L \sum_{j=1}^N y_{ik} b_x^i X_i x_j < 0
\end{cases} \]

\[ x_{qk} = \begin{cases} 
1 & \text{if } \sum_{i=1}^N x_{qik} b_y^i Y_i \geq 0 \\
-1 & \text{if } \sum_{i=1}^N x_{qik} b_y^i Y_i < 0
\end{cases} \]

where \( b \) is a positive number, \( b > 1 \); "\( \cdot \)" represents the inner product operator, \( x_{qik} \) and \( x_{qik} \) are the \( k \)th bits of \( X_q \) and the \( X_{qi} \), respectively, and \( y_k \) and \( y_{ik} \) are for \( Y \) and the \( Y_i \), respectively.

B. Energy function and Stability

Since every stored pattern pair should produce a local minimum on the energy surface [13], the energy function is intuitively defined as

\[ E(X, Y) = -\sum_{i=1}^N b_x^i X_i - \sum_{i=1}^N b_y^i Y_i. \]

Thus, the multi-e BAM network’s overall energy function is defined as

\[ E = \sum_q E(X_q, Y) = -\sum_{q=1}^L \sum_{i=1}^N (b_x^i X_i + b_y^i Y_i). \]

Assume \( E(X_{q'}, Y) \) is the energy of next state in which \( Y \) stays the same as in the previous state, and all of the other e BAM's stay at the same state as before. Hence, \( \Delta E_{q'} = -\sum_{i=1}^N b_x^i X_i - (\sum_{i=1}^N b_x^i X_i) \). Assume the \( i \)th pair is the target of the recall process for the \( q \)th e BAM. Let \( d_{x_i} \) be the Hamming distance between \( X_q \) and \( X_{q_i} \), \( d_{x_{qik}} \) the Hamming distance between the \( X_q \) and \( X_{q_i} \). Hence the \( \Delta E_{q} \) can be modified to be

\[ \Delta E_q = -\sum_{i=1}^N \log_b (b^n - 2 d_{x_{i}}) + \sum_{i=1}^N \log_b (b^n - 2 d_{x_{qik}}) \]

\[ = -\sum_{i=1}^N \sum_{k=1}^n (x_{qk} - x_{qk}) x_{qik}. \]

Note that \( \log \) is used, which is a monotonic function. From the recall process shown by (3) and (5), the \( \Delta E_{q} < 0 \) is ensured. A similar result was also given in (5) of Jeng et al. [5]. Therefore, according to Jeng’s conclusion, (3) makes \( (x_{qk} - x_{qk}) x_{qik} \) always nonnegative such that \( \Delta E_{q} \leq 0 \), and

\[ \Delta E_{q} \leq 0 \Rightarrow -\sum_{i=1}^N \log_b (b^n - 2 d_{x_{i}}) \leq -\sum_{i=1}^N \log_b (b^n - 2 d_{x_{qik}}) \]

\[ \Rightarrow -\sum_{i=1}^N b_x^i X_i - \sum_{i=1}^N b_y^i Y_i \leq -\sum_{i=1}^N b_x^i X_i - b_y^i Y_i \]

\[ \Rightarrow \Delta E_{q} \leq 0. \]

Obviously, it also holds for the other case \( E(X_q, Y') \leq E(X_q, Y) \) if the pair is heading for a stored pair, \( (X_{qi}, Y_i) \). Since the \( E(X_q, Y) \) is bounded by \( -N(b^n + b^m) \leq E(X, Y) \leq -N(b^n + b^m) \) for all \( X_q \)’s and \( Y \), the energy of the exponential BAM will converge to a stable minimum.

C. Sufficient Conditions to Recall a Stored Pattern Pair

The requirement for recalling a pair was suggested by Kosko [6], stating that a pattern pair must have the localminimal energy. As to a single e BAM, suppose the pattern pair \( (X_{q_i}, Y_i) \), \( i = 1, 2, \cdots, N \), are the stored pairs. Let the Hamming distance between \( X \) and \( X_{q_i} \) be one, which denotes the distance from the closest pattern pairs. Thus, we conclude the criteria ensured recall are

\[ -b_x^i X_i - b_y^i Y_i \geq -b_x^i X_{q_i} - b_y^i Y_i \]

\[ -b^{n-2} \geq -b^n. \]

It must be true. We can make sure that the e BAM must be guaranteed to have the correct recall according to Kosko’s criteria.

D. A Majority Rule for the Multi-e BAM Network

1) The Majority Rule of a Special Case: According to the discussion in the previous sections, every single e BAM tends to store their own pattern pairs in the local minimums of their network, respectively. Assume there are \( L \) single e BAM consisting of a multiple e BAM network, and these e BAM’s share a single output array of processing units. If these individual e BAM’s are activated by respective input patterns and they do not “agree” to have the same conclusion, i.e., the same output pattern, what will be the final result of the whole network? This problem is like a reasoning mechanism which takes any evidence into consideration at the same time to reach an optimal estimation of the hypothesis.

Hence, we formulate the entire problem as follows: Given a multi-e BAM network composed of \( L \) single e BAM’s, what is the minimal majority factor \( k, k \in [0, 1] \), to make \( kL \) e BAM’s, which are voting a common output pattern and the other e BAM’s are not, dominate the common output? In other words, we are interested in exploring the lower bound of the \( kL \) which can force the output pattern to be their common
output pattern. Note that in fact the \( kL \) denotes an integer, \( \text{Ceiling}(kL) \), which is the smallest integer larger than \( kL \). In the following text, we simply use the \( kL \) without any loss of robustness.

Before we discuss the lower bound of \( kL \), we have to study an extreme case in which a upper bound of \( kL \) will be derived. Assume the pattern pairs, \((X_{11}, Y_r), (X_{21}, Y_r), \ldots, (X_{kL}, Y_r)\), are encoded in 1st to \( kL \)th eBAM’s, respectively, and pattern pairs, \((X_{(kL+1)}, Y_r), \ldots, (X_{L}, Y_r)\), are stored in \((kL+1)\)th to \( L \)th eBAM’s, respectively. Thus, when input patterns, \( X_{11}, X_{21}, \ldots, X_{L} \), are presented at the input array of each individual eBAM, what would be the result of output?

Suppose the \( Y_r \) is the output pattern that we are looking for, i.e., it is deemed as the signal. By the SNR approach [1], [5], [13] and the evolution equations (3), we are aware of the following facts

\[
\sum_{q=1}^{L} \sum_{i=1}^{N} y_{ij} b^{X_{q,i}} X_{q,i} = \sum_{q=1}^{kL} \sum_{i=1}^{N} y_{ij} b^{X_{q,i}} X_{q,i} + \sum_{q=kL+1}^{L} \sum_{i=1}^{N} y_{ij} b^{X_{q,i}} X_{q,i}
\]

\[
= \sum_{q=1}^{kL} (y_{ij} b^n + \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i})
\]

\[
+ \sum_{q=kL+1}^{L} (y_{ij} b^n + \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i})
\]

where \( X_q \) represents the input pattern presented to the \( q \)th eBAM. The \( X_q \) can be shown in detail as follows

\[
X_q = \begin{cases} 
X_{1r}, \ldots, X_{kL} \text{ to } 1\text{st} - kL \text{th eBAM, respectively} \\
X_{(kL+1)r}, \ldots, X_{L} \text{ to } (kL+1)\text{th} - L \text{th eBAM, respectively.}
\end{cases}
\]

By our previous assumptions, only the first term in the above equation is the signal we wish to observe at the output array of processing units. As for the rest terms, they are the undesired noise. Therefore, we can derive the signal power as

\[
S = \sum_{q=1}^{kL} b^{2n} = kL b^{2n}
\]

and the largest power of noise, which means all of the rest \( (1-k)L \) eBAM’s support another output pattern \( Y_r \), is

\[
N = (1-k)L b^{2n} + kL(N-1)b^{2(n-2)} + (1-k)L(N-1)b^{2(n-2)}
\]

\[
= (1-k)L b^{2n} + L(N-1)b^{2(n-2)}.
\]

In the above noise power equation, we assume not only all of the rest \( (1-k)L \) eBAM’s support another output pattern, but also this pattern is the closest pattern to the desired one. If the desired output is intended to be recalled, then the sufficient condition is the \( S > N \) according to the SNR approach. Thus we can conclude the lower bounds for this definite recall condition of \( k \) is

\[
kL b^{2n} > (1-k)L b^{2n} + L(N-1)b^{2(n-2)}
\]

\[
k > \frac{1}{2} \frac{N-1}{N} - \frac{b^4}{2b^4}.
\]

Note that this lower bound of \( k \) means any \( k \) bigger than this threshold can force the output pattern to be the common desired output pattern of the \( kL \) eBAM’s in the network. If the bound of (7) is larger than one, however, it means even all of the eBAM’s support one output pattern, there is no guarantee to recall this common pattern.

2) The Majority Rule of the General Case: In the above extreme case, we assume all of the rest \( (1-k)L \) eBAM’s support another same output pattern which is only one bit Hamming distance away from the desired pattern. Generally speaking, however, most of the reasoning problems will not be this special. We will consider a general case in which \( kL \) eBAM’s still support a common output pattern, but the rest \( (1-k)L \) eBAM’s do not support the same output pattern, i.e., they individually support their own output patterns, respectively. Basing upon this assumption, then we can derive the following results

\[
\sum_{q=1}^{L} \sum_{i=1}^{N} y_{ij} b^{X_{q,i}} X_{q,i} = \sum_{q=1}^{kL} \sum_{i=1}^{N} y_{ij} b^{X_{q,i}} X_{q,i}
\]

\[
+ \sum_{q=kL+1}^{L} \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i}
\]

\[
= \sum_{q=1}^{kL} (y_{ij} b^n + \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i})
\]

\[
+ \sum_{q=kL+1}^{L} (y_{ij} b^n + \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i})
\]

\[
= kL \cdot y_{ij} b^n + (1-k)L \cdot y_{ij} b^n
\]

\[
+ \sum_{q=1}^{kL} y_{ij} b^{X_{q,i}} X_{q,i}
\]

\[
+ \sum_{q=kL+1}^{L} \sum_{i \neq q} y_{ij} b^{X_{q,i}} X_{q,i}
\]

where \( X_q \) represents the input pattern presented to the \( q \)th eBAM. The \( X_q \) can be shown in detail as follows

\[
X_q = \begin{cases} 
X_{1r}, \ldots, X_{kL} \text{ to } 1\text{st} - kL \text{th eBAM, respectively} \\
X_{(kL+1)r}, \ldots, X_{L} \text{ to } (kL+1)\text{th} - L \text{th eBAM, respectively.}
\end{cases}
\]

The third and fourth terms of (8) can be analyzed by the SNR approach proposed by Wang [13]. The third and fourth terms are actually sums of \( kL(N-1) \) and \( (1-k)L(N-1) \)}
independent identically distributed random variables, respectively. Therefore, the variances of the third and fourth terms are \( kL(N-1) \) and \((1-k)L(N-1)\) times the variance of a single random variable. Let

\[
\begin{align*}
  v_1 &= y_{1j}X_{q_1}X_q \\
  v_2 &= y_{2j}X_{q_2}X_q \\
  &\vdots \\
  v_N &= y_{Nj}X_{q_N}X_q.
\end{align*}
\]

Since all of the \( v_i \)'s have the same property, we select \( v_1 \) as the sample. It is trivial to derive the following probability functions for \( v_1 \)

\[
Pr(v_1 = b^{n-2-2k}) = \left(\frac{1}{2}\right)^{n-1} C_k^{n-1}
\]

\[
Pr(v_1 = -b^{n-2-2k}) = \left(\frac{1}{2}\right)^{n-1} C_k^{n-1}
\]

where the \( k \) is the Hamming distance between \( X_q \) and \( X_{q_1} \). The mean of the noise term is obviously zero. Then the variance can be derived as

\[
\begin{align*}
E[v_1^2] &= 2 \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{n-1} C_k^{n-1} \\
&= 2 \sum_{k=0}^{n-1} \left(\frac{1}{2}\right)^{m-1} C_k^{m} \\
&= 2 \left(\frac{1}{2}\right)^{m-1} (b^{-2})^m (1 + b^{-4})^{n-1} \\
&= 2 \left(\frac{1}{2}\right)^{m-1} \frac{b^{-2} + b^{-2}}{b^4} \\
&= 2 \left(\frac{1}{2}\right)^{m-1} \frac{b^{-2} + b^{-2}}{b^4}.
\end{align*}
\]

Hence, the power of the third and fourth terms are, respectively,

\[
\begin{align*}
  N_3 &= E(v_1^2) \cdot kL(N-1) \\
  N_4 &= E(v_1^2) \cdot (1-k)L(N-1).
\end{align*}
\]

The SNR of this case can be further derived

\[
SNR = \frac{kL \cdot b^{2n}}{(1-k)L \cdot b^{2n} + N_3 + N_4} \\
= \frac{k}{(1-k) + (N-1) \cdot 2\left(\frac{1}{2}\right)^{n-1} \frac{(1+b^{-4})^{n-1}}{b^4}}. \quad (11)
\]

If the common desired output pattern must be recalled, then the sufficient condition is the SNR must be greater than one. Thus, we can find the lower bound of \( k \)

\[
SNR > 1 \Rightarrow \frac{k}{(1-k) + 2^{n-1}(N-1)(1+b^{-4})^{n-1}} > \frac{1}{SNR_{eBAM}}
\]

where \( SNR_{eBAM} = \frac{2^{n-1}b^{-2}}{2(N-1)(1+b^{-4})^{n-1}} \) according to Wang’s analysis [13].

Therefore, we conclude the above discussion of a majority rule of multi-eBAM network with the following theorem

**Theorem of the Majority Rule for a Multi-eBAM Network**: Given a multi-eBAM network with \( L \) single eBAM’s, \( kL \) eBAM’s support a same common output pattern, where \( k \in [0,1] \). The condition for the output pattern of the network is the same as the one supported by the \( kL \) eBAM’s is

\[
k > \frac{1}{2} + \frac{1}{2 \cdot SNR_{eBAM}}. \quad (12)
\]

If the lower bound in (12) is larger than one, then it means that even all of the eBAM’s in the network support one output pattern, there is no guarantee to recall this output pattern.

By the above theorem, please note because the \( SNR_{eBAM} \) is usually very large, the lower bound of \( k \) can be simplified to be \( \frac{1}{2} \) which complies the human intuition. That is, if more than 50% of the evidence supports a hypothesis, then the reasoning result most likely would be the same as this hypothesis.

**E. Guaranteed Stable Condition of eBAM**

By the meaning of evolution equations of eBAM, (3), the search of the desired pattern pairs is basically a back and forth reverberation process on the energy plane, (3). Certainly, it will take more time to correctly recall a pattern pair in this way. Hence, we are interested in discovering the conditions for recalling the desired pattern pair in “one shot.” That is, we would like to discuss what the condition is to recall the correct pattern pair in one and only one back-and forth sweep. According to the first equation of (3)

\[
y_k = sgn \left( \sum_{i=1}^{N} y_{ik}b_{k}X_{iT} \right)
\]

\[
= sgn \left( b^{n}y_{Tk} + \sum_{i \neq T}^{N} y_{ik}b_{k}X_{iT} \right)
\]

where \( X_T \) is one of the stored patterns in the eBAM. Thus, if we wish to recall the pair in one shot, then the pattern pair must be stored placing in a “stable state,” i.e.,

\[
| b^{n}y_{Tk} | > \sqrt{\sum_{i \neq T}^{N} y_{ik}b_{k}X_{iT} |}
\]

Then, we have to know what the possible largest value of the right-hand side of the above equation is. The right-hand side
of the above equation can be further derived to be

\[
\left| \sum_{i \neq T} \sum_{k=1}^{N} y_{ik} b_{x_i \cdot x_T} \right| \leq \sum_{i \neq T} \left| y_{ik} b_{x_i \cdot x_T} \right| = \sum_{i \neq T} |y_{ik}| b_{x_i \cdot x_T} = \sum_{i \neq T} b_{x_i \cdot x_T} \leq \sum_{i \neq T} b^{n-2} = (N - 1)b^{n-2}.
\]

Hence, we can state that the guaranteed stable state condition for the eBAM as follows.

*Theorem of Guaranteed Stable State Condition of eBAM:* Given an eBAM, the condition for one shot correct recall of any desired pattern pair is

\[
b^2 \geq N - 1 \quad (13)
\]

where \( b \) is the base used in the eBAM evolution equations and \( N \) is the number of stored pattern pairs.

Note that the above theorem is a very strict theorem in terms of memory's capacity. It states a condition for one shot guaranteed recall of pattern pairs. Generally speaking, however, we do not demand the eBAM to recall the pattern pairs in one shot, because it will severely reduce the capacity. Hence, we would like to enlarge the capacity by relaxing the constraint, (13).

*Lemma:* Relaxation of guaranteed stable state condition by choosing a reasonable SNR, e.g., 10, is practically feasible for storing large amount of pattern pairs in eBAM.

\[
b^4 \geq SNR \cdot 2^{n+2}(N - 1) \quad (14)
\]

For example, if \( b = e, SNR = 10, n = 16 \), then we can store up to \( N = 89454 \) pattern pairs in a single eBAM.

III. SIMULATION ANALYSIS

In this section, we use some examples to illustrate the theoretical results of the multi-eBAM networks discussed in Section II.

*Example 1:* This example is used to prove the bidirectional stability of multi-eBAM networks. As shown in Fig. 1, we set \( L = 3, n = 7, p = 5, b = e \) in a multi-eBAM network. The training pattern pairs for the upper part of the network are as follows

- \( A_{11} = (1101101) \)
- \( A_{12} = (0000000) \)
- \( A_{13} = (1111111) \)
- \( A_{14} = (0100110) \)
- \( A_{21} = (1111111) \)
- \( A_{22} = (1000100) \)
- \( A_{23} = (1001101) \)
- \( A_{24} = (1100101) \)
- \( A_{31} = (0100011) \)
- \( A_{32} = (0100000) \)
- \( A_{33} = (1000001) \)
- \( A_{34} = (1001000) \)
- \( B_1 = (00101) \)
- \( B_2 = (11011) \)
- \( B_3 = (11000) \)
- \( B_4 = (00010) \)

After the pattern pairs being transformed to the bipolar mode, three input pattern are presented at respective eBAM's

- \( I_1 = (11011000) \)
- \( I_2 = A_{21} \)
- \( I_3 = A_{31} \)

The energy of the entire network according to (3) changes as follows; iteration 0, \( E = -2.38782 \times 10^3 \) iteration 1, \( E = -2.82247 \times 10^3 \) iteration 2, \( E = -3.78538 \times 10^3 \) Besides, the final pattern pairs of the network are

- \( B_{final} = B_1 \)
- \( A_{1final} = A_{11} \)
- \( A_{2final} = A_{21} \)
- \( A_{3final} = A_{31} \)

Note that the \( I_1 \) is not the same \( A_{11} \), which is one bit away from \( I_1 \). Hence, not only does it show the stability of the network, but also it shows the error correcting ability. In the following, we will prove the majority rule by presenting different combination of input patterns. If the network is given

- \( I_1 = (11011000) \)
- \( I_2 = A_{22} \)
- \( I_3 = A_{32} \).

The second and third eBAM's support the \( B_2 \) as the output pattern, but the first eBAM prefers \( B_1 \). The energy of the entire network according to (3) changes as follows; iteration 0, \( E = -2.42522 \times 10^3 \) iteration 1, \( E = -2.86787 \times 10^3 \) iteration 2, \( E = -3.81306 \times 10^3 \) The final pattern pairs of the network are

- \( B_{final} = B_2 \)
- \( A_{1final} = A_{12} \)
- \( A_{2final} = A_{22} \)
- \( A_{3final} = A_{32} \).
The result coincides with the prediction of majority rule, i.e.,
the final output pattern is \( B_2 \), because \( k = \frac{5}{3} \) is much larger
than the required minimal lower bound. Further more, if the
network is given input patterns as
\[
I_1 = ( 1 1 0 1 1 0 0 ) \\
I_2 = A_{23} \\
I_3 = A_{33}.
\]

In this case, the second and third eBAM’s support \( B_3 \) simulta-
taneously. Then, the energy of the entire network according to
(3) changes as follows: Iteration 0, \( E = -2.42789 \times 10^3 \) Iteration
1, \( E = -2.87054 \times 10^3 \) Iteration 2, \( E = -3.83341 \times 10^3 \) Besides,
the final pattern pairs of the network are
\[
B_{\text{final}} = B_3 \\
A_{1\text{final}} = A_{13} \\
A_{2\text{final}} = A_{23} \\
A_{3\text{final}} = A_{33}.
\]

According to the above result, it is obvious when more than
half of the eBAM’s in the network agree to support a common
output pattern, the output of the network will be this output
pattern.

**Example 2:** A 3-eBAM network might not be considered
to be a general case. Therefore, we construct a series of
multi-eBAM networks, \( L = 3 \) to \( L = 31 \), \( n = p = 8 \), to
verify the \( k \) prediction of the majority rule. In every multi-
eBAM simulation, the number of stored pattern pairs for each
single eBAM is also varied from \( N = 3 \) to \( N = 99 \). In this
simulation, all of the pattern pairs are randomly generated. The
result is the majority rule holds in every case, even in the worse
case, \( L = 31, N = 99 \), in which the predicted \( k = 0.515923 \),
and \( 16/31 = 0.516129 > 0.515923 \). The relation between \( k \)
and \( N \) is illustrated in Fig. 2, while the minimal \( k \) selected in
networks with different number of eBAM’s is shown in Fig. 3.

For the sake of comparison, we also repeat the simulation
for the special case, which is also deemed as a strict case. The
minimal \( k \) in this strict case is shown in Fig. 4 in which \( b = e \).
If the number of stored pattern pairs, \( N \), is larger than 55, there
is always a chance for the network to converge to the desired
common output. Because when \( N > 55 \), the \( k \) is larger than
one. According to (7), there is a chance for the network to
recall the desired common output pattern when \( N - 1 > b^2 \).
The comparison for the strict case and the general case is
shown in Table I.

**IV. CONCLUSION**

A multi-eBAM neural network has been introduced for the
belief combination in evidential reasoning. It is proved to
be bidirectionally stable, which ensures the model’s ability
to reach a local energy minimum. The sufficient conditions
for a multi-eBAM network guarantee the network to recover
a specific, predetermined pattern pair from a list of choices.
<table>
<thead>
<tr>
<th>N (number of pairs)</th>
<th>strict case</th>
<th>general case</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>0.536631</td>
<td>0.500650</td>
</tr>
<tr>
<td>10</td>
<td>0.582420</td>
<td>0.501462</td>
</tr>
<tr>
<td>15</td>
<td>0.628209</td>
<td>0.502275</td>
</tr>
<tr>
<td>20</td>
<td>0.673999</td>
<td>0.503087</td>
</tr>
<tr>
<td>25</td>
<td>0.719788</td>
<td>0.503899</td>
</tr>
<tr>
<td>30</td>
<td>0.765577</td>
<td>0.504712</td>
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<tr>
<td>35</td>
<td>0.811366</td>
<td>0.505524</td>
</tr>
<tr>
<td>40</td>
<td>0.857155</td>
<td>0.506337</td>
</tr>
<tr>
<td>45</td>
<td>0.902944</td>
<td>0.507149</td>
</tr>
<tr>
<td>50</td>
<td>0.948733</td>
<td>0.507961</td>
</tr>
<tr>
<td>55</td>
<td>0.994522</td>
<td>0.508774</td>
</tr>
<tr>
<td>60</td>
<td>1.040311</td>
<td>0.509586</td>
</tr>
<tr>
<td>65</td>
<td>1.086100</td>
<td>0.510398</td>
</tr>
<tr>
<td>70</td>
<td>1.131800</td>
<td>0.511211</td>
</tr>
<tr>
<td>75</td>
<td>1.177679</td>
<td>0.512023</td>
</tr>
<tr>
<td>80</td>
<td>1.223468</td>
<td>0.512836</td>
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<tr>
<td>85</td>
<td>1.269257</td>
<td>0.513648</td>
</tr>
<tr>
<td>90</td>
<td>1.315046</td>
<td>0.514460</td>
</tr>
<tr>
<td>95</td>
<td>1.360835</td>
<td>0.515273</td>
</tr>
</tbody>
</table>

Most important of all is the theorem of the majority rule of the network proves this neural network complies with the intuition of human reasoning. Two majority rules and their respective bounds for the majority factor, $k$, are presented. These rules will help researchers to use and predict the result of evidential reasoning. The condition for the fastest recall in the eBAM network is also discovered, which states the bound of guaranteed stable states. This network provides the ability to process many evidence at the same time reaching a consented hypothesis.

REFERENCES