An Analysis of Practical Capacity of Exponential Bidirectional Associative Memory *

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Abstract

The practical capacity of exponential bidirectional associative memory (eBAM) considering fault tolerance and fixed dynamic range of VLSI circuits is discussed. Several factors are taken into consideration in the implementation of an eBAM VLSI circuits chip. First, the fault tolerance requirement leads to the discovery of the attraction radius of the basin for each stored pattern pair. Second, the bit-error probability of the eBAM has to be optimally small when a huge amount of pattern pairs are encoded in the eBAM. Third, the fixed dynamic range of a transistor or a diode operating in the subthreshold region results in a limited length of each stored pattern. Hence, the signal-noise-ratio (SNR) analysis approach is adopted to find the attraction radius, and the practical capacity. A maximal bit-error probability (P_e) is estimated. A maximal length of patterns under a fixed dynamic range is derived.

1 Introduction

After Kosko [5] proposed the *bidirectional associative memory* (BAM), many researchers threw efforts on improving its intrinsic poor capacity and implementing the BAM with hardware circuits. For instance, among those researchers, Tai *et al.* [7] proposed a high-order BAM. However, we have pointed out that all of these improvements pay a high price of increasing the complexity of the network but only get little enhancement of the capacity [8]. Chiueh and Goodman [1] proposed an exponential Hopfield associative memory motivated by the MOS transistor's exponential drain current dependence on the gate voltage in the subthreshold region such that the VLSI implementation of an exponential function is feasible. Based upon the concept of Chiueh's exponential Hopfield associative memory, Jeng *et al.* proposed one kind of exponential BAM [4]. However, the energy function proposed in [4] can not guarantee that every stored pattern pair will have a local minimum on the energy surface. Moreover, there is no capacity analysis given in [4].

Although we have estimated the impressive capacity of an eBAM [8], it is doubtful when it comes to the hardware realization of such a neural network. There are many factors to be taken into consideration. We consider the eBAM should possess the ability of self-correcting, or called fault tolerance. In other words, we hope when a given pattern is a few bits away from a desired pattern, it should be recalled correctly. Inevitably, this property will decrease the maximal ideal capacity of the eBAM. The reason we have to discuss this fault tolerance property is that there is always some noise in a VLSI circuit. Besides, the bit-error probability of a recalled pattern is very important index to show the value of the network. Hence, the bit-error probability will determine how many patterns to be stored in an eBAM. Since Chiueh [2], Glasser [3], and Mead [6] all pointed out the dynamic range of the VLSI exponential circuits operating in the subthreshold region is approximately fixed, this property leads an important limitation in the implementation of an eBAM. That is, the dimension of the eBAM, the length of a stored pattern of the stored patterns, might be limited. In this paper, we again adopt the signal-noise-ratio (SNR) analysis approach to estimate the practical capacity of the BAM under aforementioned considerations.

2 Framework of Exponential BAM

Before proceeding the discussion of the practical capacity of the eBAM, we have to restate the framework of the eBAM neural network as a background knowledge [8].

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2.1 Evolution equations

Suppose we are given M training sample pairs, which are $\{(A_1, B_1), (A_2, B_2), ..., (A_M, B_M)\}$, where $A_i = (a_{i1}, a_{i2}, ..., a_{in})$, $B_i = (b_{i1}, b_{i2}, ..., b_{ip})$. Let X_i and Y_i be the bipolar mode of the training pattern pairs, A_i and B_i, respectively. That is, $X_i \in \{-1,1\}^n$ and $Y_i \in \{-1,1\}^p$. Instead of using Kosko's approach [5], we use the following evolution equations in the recall process of the eBAM :

$$y_k = \begin{cases} 1, & \text{if } \sum_{i=1}^{M} y_{ik} b^{X_i \cdot X} \geq 0 \\ -1, & \text{if } \sum_{i=1}^{M} y_{ik} b^{X_i \cdot X} < 0 \end{cases}, \quad x_k = \begin{cases} 1, & \text{if } \sum_{i=1}^{M} x_{ik} b^{Y_i \cdot Y} \geq 0 \\ -1, & \text{if } \sum_{i=1}^{M} x_{ik} b^{Y_i \cdot Y} < 0 \end{cases} \tag{1}$$

where b is a positive number, $b > 1$, "$\cdot$" represents the inner product operator, x_k and x_{ik} are the kth bits of X and the X_i, respectively, and y_k and y_{ik} are for Y and the Y_i, respectively. The reasons for using an exponential scheme are to enlarge the attraction radius of every stored pattern pair and to enhance the desired pattern in the recall reverberation process.

We adopt the SNR approach to compute the capacity of the exponential BAM,

$$SNR_{eBAM} = \frac{2^{n-1} b^4}{2(M-1)(1+b^{-4})^{n-1}} \tag{2}$$

where n is assumed to the $\min(n, p)$ without any loss of generality.

2.2 Attraction Radius of A Basin

In the previous subsection, the given pattern, say X, to recall the corresponding pattern pair is exactly the same as one of the stored X_i's so that the corresponding Y_i will be recalled. Suppose we initialize the eBAM with an input pattern, X, which is r bits away from the nearest pattern X_h, where $r < \frac{n}{2}$. And this input input pattern can still recall the nearest pattern and its corresponding pattern Y_h. We have the following conclusion.

Theorem 1 : Given a attraction radius, r, where $r < \frac{n}{2}$, the maximal capacity for an eBAM to store pattern pairs is

$$M < M_{\max} = 1 + \frac{2^{(n-2)}}{(n-r-1) \cdot b^{4(r-2)}} \tag{3}$$

Proof : According to Eqn.(1),

$$y_k = \text{sgn}\left\{ b^{(n-2r)} \cdot y_{hk} + \sum_{i \neq h}^{M} y_{ik} \cdot b^{X_i \cdot X} \right\} \tag{4}$$

We only discuss the Y part of evolution equations without any loss of robustness here. Assume $y_{hk} = -1$ which is the signal. The power of the signal, i.e., the first term in the right hand side of Eqn.(4) is

$$S = \left(b^{(n-2r)}\right)^2 = b^{(2n-4r)} \tag{5}$$

On the other side, the noise term of Eqn.(4) can be deemed as $(M-1)$ random variables,

$$v_1 = y_{1j} b^{X_1 \cdot X}, \ v_2 = y_{2j} b^{X_2 \cdot X}, \ \ldots, \ v_N = y_{Nj} b^{X_N \cdot X}, \tag{6}$$

where all of the $v_i, i \neq h$ are the identical random variables. We can take v_1 to compute the mean and the variance.

$$Pr(v_1 = +1 \cdot b^{n-2-2k}) = Pr(v_1 = -1 \cdot b^{n-2-2k}) = (\frac{1}{2})^{n-1} \, C_k^{n-1} \tag{7}$$

where $k \geq r+1$, and k is the Hammming distance between X and X_1. According to Eqn.(7), we certainly get $E\{v_1\} = 0$. Then, the variance can be derived as

$$E[v_1^2] \ = \ 2 \sum_{k=r+1}^{n-1} b^{2(n-2k-2)} (\frac{1}{2})^{n-1} C_k^{n-1}$$

$$= 2\sum_{k=r+1}^{m} b^{2(m-2k-1)}(\frac{1}{2})^m C_k^m, \quad \text{where} \quad m = n-1$$

$$= 2(\frac{1}{2})^m b^{2(m-1)} \sum_{k=r+1}^{m} (b^{-4})^k 1^{(m-k)} C_k^m \tag{8}$$

However, the above equation won't have a close form solution mainly because of the summation term. We can try to find the upper bound of this summation term. Let $a = b^{-4}$ to make the equations easy to read. Then, the summation term in Eqn.(8) can be rewritten as

$$\begin{aligned}\sum_{k=r+1}^{m} a^k \cdot C_k^m &= \sum_{k=0}^{m} a^k \cdot C_k^m - \sum_{k=0}^{r} a^k \cdot C_k^m = (1+a)^m - \sum_{k=0}^{r} a^k \cdot \frac{m!}{(m-k)! \cdot k!} \\ &= (1+a)^m - \sum_{k=0}^{r} a^k \cdot \left(\frac{m}{m-k}\right)\left(\frac{m-1}{m-k-1}\right)\cdots\left(\frac{r+1}{r+1-k}\right) \cdot \frac{r!}{(r-k)! \cdot k!} \\ &< (1+a)^m - \sum_{k=0}^{r} a^k \cdot C_k^r = (1+a)^m - (1+a)^r\end{aligned}$$

Hence, Eqn.(8) can be derived as

$$\begin{aligned}E[v_1^2] &< 2(\frac{1}{2})^m b^{2(m-1)} \left[(1+b^{-4})^m - (1+b^{-4})^r\right] \\ &= 2(\frac{1}{2})^{(n-1)} b^{2(n-2)} \left[(1+b^{-4})^{(n-1)} - (1+b^{-4})^r\right]\end{aligned} \tag{9}$$

The above upper bound is the maximal noise power, called $N_{\max}$. Then the minimal signal-noise-ratio ($SNR_{\min}$) of the eBAM is

$$\begin{aligned}SNR_{\min} &= \frac{S}{(M-1)N_{\max}} = \frac{b^{(2n-4r)}}{(M-1) \cdot 2(\frac{1}{2})^{(n-1)} b^{2(n-2)} \left[(1+b^{-4})^{(n-1)} - (1+b^{-4})^r\right]} \\ &= \frac{b^4 \cdot 2^n}{4(M-1) \cdot b^{4r} \cdot \left[(1+b^{-4})^{(n-1)} - (1+b^{-4})^r\right]}\end{aligned} \tag{10}$$

The $SNR_{\min}$ must be greater than 1 in order to recall the correct pattern pair. Hence, $SNR_{\min} > 1$,

$$\begin{aligned}b^4 \cdot 2^n &> 4(M-1) \cdot b^{4r} \cdot \left[(1+b^{-4})^{(n-1)} - (1+b^{-4})^r\right] \\ &\approx 4(M-1) \cdot b^{4r} \cdot \left[(1+(n-1) \cdot b^{-4}) - (1+r \cdot b^{-4})\right] \\ &= 4(M-1)(n-r-1) \cdot b^{4(r-1)} \\ 2^n &> 4(M-1)(n-r-1) \cdot b^{4(r-2)} \\ M &< 1 + \frac{2^{(n-2)}}{(n-r-1) \cdot b^{4(r-2)}} = M_{\max}\end{aligned} \tag{11}$$

The attraction radius, in other words, can be deemed as the ability of fault tolerance of the eBAM. Basically, if we wish the fault tolerance ability of an eBAM to be enhanced, then unavoidably the Hamming distance between every two stored pattern pairs must be increased so that the amount of pattern pairs to be stored will be decreased. Referring to Fig. 1, it shows the capacity of an eBAM with fault tolerance ability will drastically decrease with the increase of the radius of the attraction basin of each stored pattern pair.

Consequently, the amount of stored pattern pairs certainly affects the probability of correct recall. It is very important to realize whether the amount of stored pattern pairs exceeds a predetermined tolerable error probability. Any memory with poor accuracy is worthless in any regards. The central limit theorem is employed to develop an estimation of the bit-error probability of an eBAM given certain amount of stored pattern pairs.

Theorem 2 : The bit-error probability of the eBAM is

$$P_e \approx \frac{1}{\sqrt{2\pi}} \cdot \frac{(M-1)^{\frac{1}{2}} 2^{(-\frac{n}{2}+1)}(n-r-1)^{\frac{1}{2}}}{b^{(4-2r)}} \cdot \exp\left[-\frac{b^{(8-4r)}}{2(M-1)\cdot 2^{(-n+2)}\cdot(n-r-1)}\right] \tag{12}$$

Proof : y_{hk} can be assumed to be -1 without loss of robustness. Then, the error occurs when the argument in the sgn function of Eqn.(4) turns out to greater than 0. That is,

$$\begin{aligned} P_e &= Prob(V > 0), \\ V &= -b^{n-2r} + \sum_{i \neq h}^{M} y_{ik} \cdot b^{X_i \cdot X} = -b^{n-2r} + v \end{aligned}$$

Again we assume the summation term in the above equation is the sum of $(M-1)$ identical random variables as described in the proof of Theorem 1. We also take v_1 as an example to compute the expectation value and the variance of each random variable. According to Eqn.(9),

$$\begin{aligned} Var\{v_1\} &= E\{v_1^2\} - E^2\{v_1\} \\ \sigma_{v_1} &= \sqrt{Var\{v_1\}} < \sqrt{2(\frac{1}{2})^{(n-1)} b^{2(n-2)} \left[(1+b^{-4})^{(n-1)} - (1+b^{-4})^r\right]} \\ &\approx 2^{(-\frac{n}{2}+1)} b^{(n-2)} \sqrt{(n-r-1)\cdot b^{-4}} \\ &= 2^{(-\frac{n}{2}+1)} b^{(n-4)} (n-r-1)^{\frac{1}{2}} \end{aligned}$$

Since the overall noise, v, is the sum of $(M-1)$ identical random variables, e.g., v_1, we surely have the following conclusion.

$$Var\{v\} = (M-1)\cdot Var\{v_1\}, \quad \sigma_v = (M-1)^{\frac{1}{2}} \sigma_{v_1} \tag{13}$$

Henceforth, the error of the recall process occurs when $v > 0$ in Eqn.(4). Basing upon the central limit theorem, we should have the following result when $n \to \infty$, $M \to \infty$.

$$P_e = Prob(V > 0) = Prob(v > b^{n-2r}) = Q\left(\frac{b^{n-2r}}{\sigma_v}\right) \tag{14}$$

where

$$Q(t) = \frac{1}{\sqrt{2\pi}} \int_t^{\infty} e^{-\frac{x^2}{2}} dx \approx \frac{1}{\sqrt{2\pi}} \cdot t^{-1} \cdot e^{-\frac{t^2}{2}} \tag{15}$$

Hence, the bit-error probability P_e is defined as

$$P_e \approx \frac{1}{\sqrt{2\pi}} \cdot \frac{(M-1)^{\frac{1}{2}} 2^{(-\frac{n}{2}+1)}(n-r-1)^{\frac{1}{2}}}{b^{(4-2r)}} \cdot \exp\left[-\frac{b^{(8-4r)}}{2(M-1)\cdot 2^{(-n+2)}\cdot(n-r-1)}\right] \tag{16}$$

Though the above equation looks very complicated at first glance, a comprehensive result is given is Fig. 2 which shows the relationship between P_e and M, n, r. Another astonishing result given in Lemma 1 showing that when the largest amount of stored pattern pairs under a given attraction radius is stored in an eBAM, the bit-error probability is a constant which is independent from the dimension of the eBAM.

Lemma 1 : When the number of the stored pattern pairs is approaching the upper bound of a given attraction radius, as defined in Theorem 1, the bit-error probability will approach a fixed value. That is,

$$P_e \approx \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}) \approx 0.24197, \qquad \text{when } M = M_{\max} \tag{17}$$

Proof : According to Eqn.(11), if $M = M_{\max}$, then

$$\begin{aligned} \exp\left[-\frac{b^{(8-4r)}}{2(M-1)\cdot 2^{(-n+2)}\cdot(n-r-1)}\right] &= \exp\left[-\frac{b^{(8-4r)}}{2\cdot\frac{2^{(n-2)}}{(n-r-1)\cdot b^{4(r-2)}}\cdot 2^{(-n+2)}\cdot(n-r-1)}\right] \\ &= \exp\left[-\frac{b^{4(r-2)}\cdot b^{(8-4r)}}{2}\right] = \exp(-\frac{1}{2}) \end{aligned}$$

and

$$\frac{(M-1)^{\frac{1}{2}}2^{(-\frac{n}{2}+1)}(n-r-1)^{\frac{1}{2}}}{b^{(4-2r)}} = \frac{(\frac{2^{(n-2)}}{(n-r-1)\cdot b^{4(r-2)}})^{\frac{1}{2}}2^{(-\frac{n}{2}+1)}(n-r-1)^{\frac{1}{2}}}{b^{(4-2r)}} = 1$$

Thus, we have the following conclusion,

$$P_e \approx \frac{1}{\sqrt{2\pi}} \cdot \exp(-\frac{1}{2}) \approx 0.24197 \tag{18}$$

2.3 Dynamic Range Limitation

Chiueh [2], Glasser [3], and Mead [6] pointed out that a transistor operating in the subthreshold region working as an exponentiation circuit has a dynamic range of approximate 10^5 to 10^7. In the realization of eBAM by VLSI circuit, hence, the dimension, n, and the amount of stored pattern pairs, M, must also be limited. Assume the dynamic range D to be $D = b^n$. When the dimension of the eBAM, n, increases, then b will decrease. Thus, the capacity given in Eqn.(3) should be changed and recalculated accordingly. Referring to Fig. 3, the increase of radix b will cause the drastical decrease of the dimension of the eBAM. An estimation of the practical capacity of an eBAM under this dynamic range limitation is shown in the following discussion.

Theorem 3 : Given a fixed dynamic range, $D = b^n$, the maximal number of pattern pairs allowed to be stored in an eBAM with dimension n, is

$$M \approx \frac{D^{\log_b 2}}{\log_b D} \tag{19}$$

Proof : Since $b > 1$ and D is a fixed number, we'd like to let b as small as possible in order to get a large dimension, n. Hence, assume $b = 1 + \mu$, where μ is a small positive number.

$$D = b^n, \quad \log D = n \log b = n \log(1+\mu) \approx n \cdot \mu \tag{20}$$

Basing upon Eqn.(11), if $\mu \to 0, n \to \infty$, the $M_{\max}$ can be rewritten as

$$M_{\max} \approx \frac{2^{(n-2)}}{(n-r-1)\cdot b^{4(r-2)}} \approx \frac{2^{n-2}}{[1+4(r-2)\mu]\cdot n} \approx \frac{2^{n-2}}{n} \approx \frac{2^n}{n} \tag{21}$$

Thus, we can have the following approximation equation, $M_{\max} \approx \frac{D^{\log_b 2}}{\log_b D}$.

The dimension of the eBAM, thus, will be restricted by the fixed dynamic range. A proper size of the eBAM if e is used as the radix is $n = p = 16$ accordingly.

3 Simulation Analysis

Example 1. In order to verify the prediction of the attraction radius and the bit-error probability, we have conducted simulations to confirm the practical capacity of the eBAM. Referring to Table 1, a 16×16 eBAM with randomly generated stored pattern pairs has undergone the simulation of recalling process of pattern pairs.

M	r	No. of test	No. of failures	error rate	theoretical P_e
1300	1	15180	0	0.0000	≈ 0
1500	1	17122	1	0.0001	≈ 0
1600	1	17570	6	0.0003	≈ 0
1800	1	18452	9	0.0005	≈ 0
100	2	10078	0	0.0000	0.0002
500	2	22143	224	0.0100	0.0710
1000	2	14712	304	0.0200	0.1890
1200	2	11240	823	0.0730	0.2300
1500	2	6839	948	0.1386	0.2858

Table 1: The simulation of the 16×16 eBAM.

Note that the recalling ability of the eBAM under practical consideration is still very outstanding and promising. The P_e in Eqn.(12) is proved to be a good approximation of the upper bound of the bit-error probability. The The above data for $r = 2$ is also shown in Fig. 4 to verify the prediction of P_e.

4 Conclusion

We have discussed the practical capacity of the eBAM when it comes to the consideration of its hardware realization. The feasible size of an eBAM implemented with VLSI circuits is also estimated. The simulation results turn out to be coherent with the prediction of the theory. In short, the eBAM with fault tolerance ability is worthy of implementation by VLSI circuits.

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Fig. 1. The practical capacity of eBAM. (b=c)

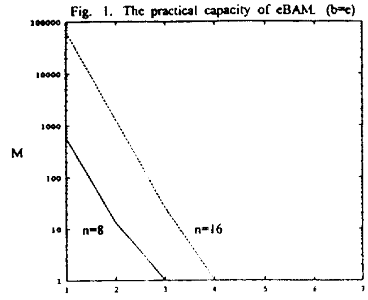

Fig. 2. The bit-error probability. (n=p=16)

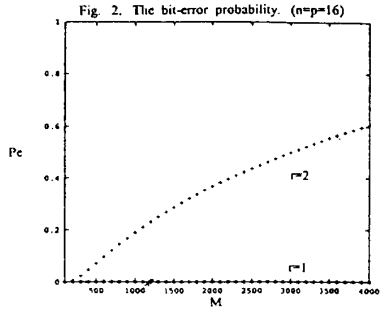

Fig. 3. Dynamic range limitation.

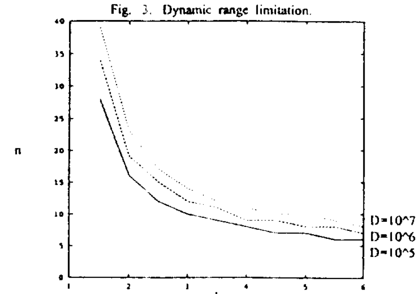

Fig. 4. The bit-error probability of eBAM.